Canonical bundle formula and degenerating families of volume forms

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Based on

- Canonical bundle formula and degenerating families of volume forms, arXiv.
- L2 extension of holomorphic functions for log canonical pairs, arXiv.

Volume asymptotics

- Let $f : X^{m+n} \to Y^m$ be a degenerating family of compact complex manifolds (of dim n).
- Let Ω_t be a "family" of holomorphic n forms on the fiber X_t, $t \in Y$.

What is

$$v(t) := \int_{X_t} |\Omega_t|^2 = \int_{X_t} c_n \Omega_t \wedge \overline{\Omega_t}$$

as a function of t?

- Problem: Identify the asymptotics/singularity ("poles and zeros") of $v(t)$, i.e. identify $v(t)$ up to a bounded factor.
- Equivalently **Asymptotics of L^2 metrics** (to be defined) in a general geometric setting.

Geometric setting: Log Calabi-Yau fibrations

- More precisely, let $f : X^{m+n} \to Y^m$ be a surjective projective morphism with connected fibers between complex manifolds.

A pair (f, L) of $f : X \to Y$ and a line bundle L is **Log Calabi-Yau** if $K_X + L$ is the pullback of "some" line bundle, i.e. there exists a line bundle M on Y such that

$$K_X + L = f^*(K_Y + M)$$

holds. (K_X, K_Y canonical line bundles)

- Additive notation of line bundles : $L_1 + L_2 := L_1 \otimes L_2$
- If $L = \mathcal{O}_X$, a general smooth fiber X_t has K_{X_t} trivial.

A special case

Example (Kodaira 1963, ..., T. Fujita 1986)

- **Elliptic fibration** $f : X^2 \to Y^1$: a general fiber X_t is an elliptic curve.
- Singular fibers are classified (Kodaira):
  $f^{-1}(P_i) \in [{}_mI_b, II, III, IV, I^*, II^*, III^*, IV^*]$, $a_i \in [1 - \frac{1}{m}, \frac{1}{2}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}]$
- Canonical bundle formula relates K_X and K_Y :

$$K_X = f^*(K_Y + (\textit{moduli part}) + \sum a_i P_i)$$

as equality of $\mathbf{Q}$-line bundles; (moduli part) $= \frac{1}{12} j^* \mathcal{O}_{\mathbf{P}_1}(1)$, $j : Y \to \mathbf{P}^1$.

Line bundles and Singular hermitian metrics

- Let L be a holomorphic line bundle on a complex manifold X with transition functions $\{g_{ij}\}$ on a locally trivializing open cover $\{U_i\}_{i\in I}$.
- A (smooth or singular) hermitian metric $e^{-\varphi}$ of L can be identified with a family of functions $e^{-\varphi_i}$ satisfying $e^{-\varphi_i} = |g_{ij}|^{-2}e^{-\varphi_j}$ for evey $i, j \in I$.
- Abuse of notation : $e^{-\varphi}$ refers to the collection of $e^{-\varphi_i}$'s
- We define $e^{-\varphi}$ to be a **singular hermitian metric** if $\forall i$, φ_i is plurisubharmonic.

Plurisubharmonic functions

Definition

Let U be an open subset of $\mathbf{C}^n$. Let $\varphi : U \to \mathbf{R} \cup \{-\infty\}$ be a function, not $\equiv -\infty$ on any connected component of U. The function φ is **plurisubharmonic** (**psh** for short) if

1. φ is upper semicontinuous, and
2. for any complex line L in $\mathbf{C}^n$, the restriction of φ to $L \cap U$ is either subharmonic or $\equiv -\infty$ (on each connected component of $L \cap U$).

- Example: $\varphi(z) = \log|g(z)|$ is psh when g is holomorphic.
- Whenever $\varphi(z) = -\infty$ (or φ is not locally bounded), we will say that φ has singularities.
- In general, a psh function can have complicated singularities.

- Let $a \in U$. The **Lelong number** of φ at a is defined as

$$\nu_a(\varphi) := \max\{\gamma \geq 0 : \varphi(z) \leq \gamma \log|z-a| + O(1) \text{ near } a\ \}$$

- Curvature current $i\partial\overline{\partial}\varphi := i\partial\overline{\partial}\varphi_j$ is well-defined and "nonnegative" as a current of bidegree (1,1).

Example

Let $s \in H^0(X, L)$ be a holomorphic section, viewed as a family of holomorphic functions s_i satisfying $s_i = g_{ij}s_j$ on $\{U_i\}$. Taking $\varphi_i = \log|s_i|^2$, it defines a psh metric of L. The curvature current is the one associated to the divisor of s.

- A singular hermitian metric (or a **psh metric**) of L generalizes a holomorphic section of L in this sense.

L^2 metric of a log Calabi-Yau fibration

- Let $f : X^{m+n} \to Y^m$ be a log Calabi-Yau fibration.
- Since $K_X + L = f^*(K_Y + M)$, the projection formula says $M = f_*(K_{X/Y} + L)$ where $K_{X/Y} := K_X - f^*(K_Y)$, the relative canonical line bundle.
- A holo. section s of the direct image $f_*(K_{X/Y} + L) = M$ defines a family of $(L|_{X_t})$-valued n-forms $t \mapsto \sigma_t = \sigma|_{X_t}$ $(t \in Y)$.
- Remark: "σ" is not a globally (uniquely) defined, but with a choice of $t = (t_1, \ldots, t_m)$, $\sigma \wedge dt$ is globally defined $(dt = dt_1 \wedge \ldots \wedge dt_m)$.
- $\sigma \wedge dt = \tilde{\sigma} \wedge dt$ if and only if $\sigma|_{X_t} = \tilde{\sigma}|_{X_t}, \forall t$

L^2 metric of a log Calabi-Yau fibration

- A holo. section s of the direct image $f_*(K_{X/Y}+L) = M$ defines a family of $(L|_{X_t})$-valued n-forms $t \mapsto \sigma_t = \sigma|_{X_t}$ $(t \in Y)$.
- Now let $e^{-\lambda}$ be a sing. herm. metric of L.
- The induced L^2 metric $e^{-\mu}$ for M is defined by the **fiberwise integration** : for X_t smooth fibers,

$$\left(|s|^2 \cdot e^{-\mu}\right)(t) = \int_{X_t} i^{n^2} \sigma|_{X_t} \wedge \overline{\sigma}|_{X_t} e^{-\lambda}$$

- Can also view the **fiberwise integration** as an operator sending $2(n+m)$-forms on $X \to 2m$-forms on Y.
- **Volume Asymptotics**

  = Asymptotics (Singularities) of L^2 metrics

  = Asymptotics of the fiberwise integration
- More generally, we can allow L, M to be $\mathbf{Q}$-line bundles.

Volume asymptotics for dim $Y = 1$

- Asymptotics for $e^{-\mu}$ in $\left(|s|^2 \cdot e^{-\mu}\right)(t) = \int_{X_t} i^{n^2} \sigma|_{X_t} \wedge \overline{\sigma}|_{X_t} e^{-\lambda}$ can be roughly summarized as (t is a local coord. on Y) :

$$(|t|^2)^a \cdot |\log|t|^2|^b$$

- Recent: [Yoshikawa 2010], [Takayama 2016], [Gross-Tosatti-Zhang 2016], [Berman 2016], [Boucksom-Jonsson 2017], [Eriksson-Freixas i Montplet-Mourougane 2018]
- Old: [Arnold-Gusein-Zade-Varchenko 1984, Theorem 10.2] and more
- Various settings and statements : Equality or upper bound, Log version (twisted by L) or not, Information on the exponent a, ...

Volume asymptotics for dim $Y = 1$

- Let $f : X^{n+1} \to Y^1$ be a surjective projective morphism between complex manifolds where dim $Y = 1$.
- Assume $L = 0$ i.e. $K_X + 0 = f^*(K_Y + M)$ so that $M = f_* K_{X/Y}$ (and a general smooth fiber of f has trivial canonical line bundle).
- Suppose that $X_0 = \sum a_j E_j$ snc (simple normal crossing).

Theorem (Eriksson-Freixas-Mourougane '18 cf. Boucksom-Jonsson '17, cf. Takayama '16, ...)

The L^2 metric for $M := f_* K_{X/Y}$ has the asymptotics (near $t = 0$)

$$e^{-\mu} = \left(\frac{1}{|t|^2}\right)^{(1-c(X_0))} |\log|t|^2|^b$$

where t is a local holomorphic coordinate on Y, $b \geq 0$ is an integer and $c(X_0)$ is the log-canonical threshold of (X, X_0).

- The log-canonical threshold of a pair $(X, D = (g = 0))$ is the supremum of the exponent c for which $\frac{1}{|g|^{2c}}$ is locally integrable.

Volume asymptotics for dim $Y \geq 2$

Question

Generalization for higher dimension of Y?

- The only previous result: as a consequence of Hodge theory [Cattani-Kaplan-Schmid 1986], Kawamata's semipositivity theorem has a case when the L^2 metric has vanishing Lelong numbers ($\sim |\log|t|^2|^b$).
- Difficulty in the general case : What will be generalization of $(|t|^2)^a \cdot |\log|t|^2|^b$?
- There is a candidate for generalization of $|t|^{2a}$ from algebraic geometry, defined in the general canonical bundle formula due to Kawamata.
- Starting from a general $f : X \to Y$, one can use Hironaka's resolution both on X and on Y to come to the following situation. ("Resolution of f")

Conditions for SNC fibration: “Resolution of f”

- X, Y are complex manifolds.
- $B = \sum B_i$ is a "reduced" snc divisor on Y (i.e. $B = \mathrm{red}(B)$).
- $R = R_h + R_v$ is an snc divisor on X with $f(\mathrm{Supp}(R_v)) \subset \mathrm{Supp}\, B$.
- $\mathrm{red}(R) + f^*B$ is an snc divisor on X.
- f is a smooth morphism over $Y \setminus B$; R_h is a relative snc divisor over $Y \setminus B$.
- $K_X + R$ is $\mathbf{Q}$-linearly equivalent to the pullback of some $\mathbf{Q}$-line bundle on Y (which we can write as $K_Y + M$, thus an LCY fibration).
- We will call $f : (X, R) \to (Y, B)$ an SNC LCY fibration. (simple normal crossing + log Calabi-Yau)
- Such f is locally given by monomials in local coordinates.

Definition (Kawamata 1998)
Given an SNC LCY fibration $f : (X, R) \to (Y, B)$ (in particular given R), define the **discriminant divisor** of R to be

$$B_R := \sum c_i B_i$$

where the coefficient c_i is given by log-canonical thresholds :

$$1 - \sup\{c : (X, R + cf^*B_i) \text{ is log-canonical over the generic point of } B_i\}$$

- B_R is supposed to capture singularities of fibers and the divisor R.
- B_R generalizes Kodaira's $\sum a_i P_i$ from the elliptic fibration case.
- Kawamata said "The coefficients were defined so that they behave well under semi-stable reduction."

- The **moduli part line bundle** $J := J(X/Y, R)$ is defined to be the $\mathbf{Q}$-line bundle satisfying the equality of $\mathbf{Q}$-line bundles: $K_X + L = \mathcal{O}(K_X + R) = f^*(K_Y + M) = f^*(\mathcal{O}(K_Y + B_R) + J)$.

Theorem (Kawamata's general canonical bundle formula)

From $K_X + R = f^*(K_Y + B_R + J)$, the moduli part line bundle J is nef.

- This generalizes the elliptic fibration case where J was $\frac{1}{12} j^* \mathcal{O}_{\mathbf{P}_1}(1)$, $j : Y \to \mathbf{P}^1$ (thus semiample).
- This is applied for "Subadjunction of log-canonical centers" in [Kawamata '98]. ($\to$ applications to MMP)

Question 1: cf. [Eriksson-Freixas-Mourougane]

Let $f : (X, R) \to (Y, B)$ be an SNC log Calabi-Yau fibration.

- Is there a **metric version/approach** for this general canonical bundle formula of [Kawamata 1998]?
- More precisely, does the L^2 **metric** (induced by the divisor R) have singularities described by the **discriminant divisor** B_R ? (up to an extra psh weight with vanishing Lelong numbers)

- Recall: $K_X + L = \mathcal{O}(K_X + R) = f^*(K_Y + M) = f^*(\mathcal{O}(K_Y + B_R) + J)$

- Let $f : (X, R) \to (Y, B)$ be an SNC log Calabi-Yau fibration (with $f_0 : X_0 \to Y_0$ the smooth fibers).
- Let α be a singular volume form on X with poles along $R = \sum_{i=1}^{k} a_i R_i$, $R_i = \mathrm{div}(w_i)$ as in (up to some smooth bounded factor)

$$\alpha(w) = (\prod_{i=1}^{k} |w_i|^{-2a_i}) |dw_1 \wedge \ldots \wedge dw_n|^2$$

Question 2 : Analytic characterization of the discriminant divisor

When we integrate α along fibers of f_0, do we get a singular volume form on Y having poles along the discriminant divisor B_R (times a psh weight $e^{-\varphi}$ with vanishing Lelong numbers) as in

$$(f_*\alpha)(z) = (\prod_{j=1}^{p} |z_j|^{-2c_j}) e^{-\varphi(z)} |dz_1 \wedge \ldots \wedge dz_m|^2 \text{ ?}$$

- Our original motivation was its application to L^2 **extension theorems**
- Question 1 implies Question 2 (essentially equivalent, but Q1 slightly more precise).
- Subtlety: a priori, the "extra factor" $e^{-\varphi(z)}$ may have nothing to do with semipositive curvature (i.e. psh).

Main result

- Question 1 and Question 2 have positive answers.

Theorem (K.)

- *Let* $f : (X, R) \to (Y, B)$ *be an SNC log Calabi-Yau fibration :*
  $K_X + R = f^*(K_Y + B_R + J)$ *: in terms of line bundles,*
  $K_X + L = f^*(K_Y + M) := f^*(K_Y + J + H)$ *where* $H = \mathcal{O}(B_R)$
- *Let* λ *be a sing herm metric of* L *given by the snc divisor* R *(not necess. effective).*
- *Then the* **induced L^2 metric** μ *for the* $\mathbf{Q}$*-line bundle* M *is equal to the product of sing. herm. metrics* (H, η) *and* (J, ψ)*, i.e.*

$$e^{-\mu} = e^{-\eta} e^{-\psi} \text{ where}$$

- η *is a sing. herm. metric given by the* **discriminant divisor** B_R *and*
- ψ *is a sing. herm. metric of* J *with nonnegative curvature current (i.e. a psh metric) and with vanishing Lelong numbers.*

- In particular, the moduli part J is nef.

Sketch of proof

- **Step 1**: As an important example, first **artificially** assume that the fiber integral $f_*\alpha$ of α is known to (*) have poles along some snc divisor Γ on Y (ignoring $e^{-\varphi}$ with vanishing Lelong numbers, for the moment).
- Then using $f_*(\alpha \wedge f^*t) = f_*\alpha \wedge t$ for appropriate real-valued functions of the type $t = \prod_{i=1}^{k} |w_i|^{2b_i}$, we can show that Γ must be equal to the discriminant divisor B_R.
- Take t with poles along $\delta B - \Gamma \ \ (\delta < 1)$. Then $f_*\alpha \wedge t$ is locally integrable, thus so is $\alpha \wedge f^*t$. This implies $\Gamma \geq B_R$.
- Suppose that $\Gamma \neq B_R$. One gets contradiction by taking t with poles along $\delta B - B_R$.

Sketch of proof

- **Step 2(??)**: Now try to show the condition (*) (this time together with $e^{-\varphi}$) by direct computation, which will proceed as follows.
- Since $f : (X, R) \to (Y, B)$ is SNC, it is given by monomials in local coordinates w on X and z on Y adapted to the snc divisors R and B.
- For $i = 1, \ldots, m$, the map f is given by $z_i = w_1^{a_{1i}} \ldots w_{n+m}^{a_{(n+m)i}}$.
- We need to integrate on each smooth fiber, $\alpha = \frac{1}{\prod |w_i|^{2r_i}} |dw_1 \wedge \ldots \wedge dw_{n+m}|^2$ which is then written like $\alpha = \frac{1}{\prod_{a_i \neq 0} |a_i/w_i|^2 \prod |w_i|^{2r_i}} |f^* \frac{dz_1}{z_1} \wedge \cdots \wedge f^* \frac{dz_m}{z_m}|^2 \wedge |dw_{1+m} \wedge \cdots \wedge dw_{n+m}|^2$. where $w_{1+m}, \ldots, w_{n+m}$ are fiber variables.

- $\alpha = \frac{1}{\prod_{a_i \neq 0} |a_i / w_i|^2 \prod |w_i|^{2r_i}} |f^* \frac{dz_1}{z_1} \wedge \cdots \wedge f^* \frac{dz_m}{z_m}|^2 \wedge |dw_{1+m} \wedge \cdots \wedge dw_{n+m}|^2.$
  where $w_{1+m}, \ldots, w_{n+m}$ are fiber variables.
- Use $w_{m+j} = e^{\rho_j} e^{i\theta_j}$ for $j = 1, \ldots, v$ for some $v \leq n$.
- Eventually it reduces to computing
  $$\int e^{c_v \rho_v} \ldots \int e^{c_2 \rho_2} \left(\int e^{c_1 \rho_1} d\rho_1 \right) d\rho_2 \ldots d\rho_v$$
  where the intervals for the repeated integration are as follows:
- $0 \geq \rho_v \geq \max \left(b_{v1}^{-1} \log|z_1|, \ldots, b_{vm}^{-1} \log|z_m| \right)$
  $0 \geq \rho_{v-1} \geq \max \left(b_{v-1,1}^{-1} (\log|z_1| - b_{v1}\rho_v), \ldots, b_{v-1,m}^{-1} (\log|z_m| - b_{vm}\rho_v) \right)$
  ...etc
  (where any item in the max involving b_{ji}^{-1} with $b_{ji} = 0$ should be replaced by $-\infty$.)
- At this point, subtracting some divisor part Γ, we need to check **plurisubharmonicity** AND **vanishing Lelong numbers** by hand, which seems practically impossible.
- **Instead**, we will use some outside input of plurisubharmonicity !

Positivity of direct images

Theorem (Păun-Takayama)

Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Let L be a line bundle on X such that $K_X + L = f^*(K_Y + M)$ for some line bundle M on Y. Suppose that (L, λ) is a psh metric and that the inclusion

$$f_*(K_{X/Y} \otimes L \otimes \mathcal{J}(\lambda)) \to f_*(K_{X/Y} \otimes L)$$

*is generically an isomorphism. Then the induced L^2 **metric** μ for M is a **psh metric**.*

- Proof based on Hörmander L^2 estimates.
- Idea: extend the L^2 metric for $f_0 : X_0 \to Y_0$ from Y_0 to Y as a psh metric.

Sketch of proof

- Thanks to the previous theorem [PT], the induced L^2 metric is a psh metric. Since the L^2 metric is characterized by the fiber integral, we now know that the fiber integral looks like

  $$(f_*\alpha)(z) = e^{-\mu(z)}|dz_1 \wedge \ldots \wedge dz_m|^2$$

  for some psh function μ.
- However, the psh singularity of μ can be extremely complicated, far from a nice algebraic one given by an snc divisor !!
- How can we show that it actually looks like (for φ with vanishing Lelong numbers)

  $$(f_*\alpha)(z) = \left(\prod_{j=1}^{p}|z_j|^{2c_j}\right)e^{-\varphi(z)}|dz_1 \wedge \ldots \wedge dz_m|^2 \text{ ?}$$
- We will use the **valuative viewpoint** for plurisubharmonic singularities, which makes up for the lack of a log-resolution by considering all divisorial valuations (over the variety Y), i.e. all exceptional divisors which can appear after repeated blow-ups over Y.

Equivalence of PSH singularities

- Let u and v be psh functions on a complex manifold X.

- We will say u is more singular than v if $u \leq v + O(1)$ i.e. $u - v$ is locally bounded above.
- When $u \leq v + O(1)$ and $v \leq u + O(1)$, we will say u and v have **equivalent singularities** and write $u \sim v$.

- When $u \sim v$, they have all the same 'measures' of singularities : Lelong numbers, multiplier ideals, log canonical thresholds, jumping numbers, higher Lelong numbers,...
- Hence for the purpose of psh singularity, we can often consider psh functions up to this equivalence.

Example

Let $f_1, \ldots, f_m$ be (local) holomorphic functions.

$$\log \sum |f_i| \sim \frac{1}{2} \log \sum |f_i|^2 \sim \log \max_i |f_i|$$

Valuative equivalence of PSH singularities

- The **multiplier ideal sheaf** of a psh function u on X is the ideal sheaf $\mathcal{J}(u)$ of germs $f \in \mathcal{O}_X$ locally satisfying

$$\int |f|^2 e^{-2u} dV < \infty.$$

Definition

Two psh functions φ and ψ on Y are **valuatively equivalent** if the following two equivalent conditions (thanks to [Boucksom-Favre-Jonsson 2008] and the openness theorem of [Guan-Zhou]) hold:

- (1) For all real $m > 0$, **all the multiplier ideals are equal** : $\mathcal{J}(m\varphi) = \mathcal{J}(m\psi)$.
- (2) At every point of all proper modifications over X, the Lelong numbers of φ and ψ coincide. In other words, **for every divisorial valuation v** centered on Y, we have $v(\varphi) = v(\psi)$. (Generic Lelong number along the (exceptional) divisor E where $v = v_E$.)

Sketch of proof

- Now we go back to the proof.
- **Step 1**: Assuming that the fiber integral $f_*\alpha$ of α is already known to have poles along some snc divisor Γ on Y, we were able to show that Γ must be equal to the discriminant divisor B_R.
- **Step 2**: We know that the fiber integral looks like
  $$(f_*\alpha)(z) = e^{-\mu(z)}|dz_1 \wedge \ldots \wedge dz_m|^2$$
  for some psh function μ.
- Now we show that μ is valuatively equivalent to the psh function ψ_{B_R} which is associated to the discriminant divisor B_R (which we can assume effective from assuming R effective) : $v(\mu) = v(\psi_{B_R})$.
- The proof is adaptation of the argument in **Step 1** to a higher model $\pi : Y' \to Y$ such that v is realized as a prime divisor in Y'.

Sketch of proof

- **Step 3**: μ is a psh metric for the line bundle $M := H + J$ while ψ_{B_R} is for $H := \mathcal{O}(B_R)$. From the Siu decomposition of the curvature current of μ, $\Theta_\mu = \sum \nu(T, Y_k)[Y_k] + R_T$, the divisor part belongs to the first chern class of H.
- R_T is with vanishing Lelong numbers and belongs to J. There exists a sing. herm. metric for J with the curvature current equal to R_T.
- Since we identified the "full psh singularity" to be precisely equal to the discriminant divisor, the moduli part line bundle is left with "empty singularity".

What is L^2 Extension?

- Let $Y \subset X$ be a submanifold of a complex manifold. Let L be a line bundle on X and K_X the canonical line bundle of X.
- Very roughly, an *L^2 extension theorem* is a statement that (under suitable conditions on $X, Y, L, \ldots$):

  If a certain L^2 norm $\|s\|_Y$ is finite for a holomorphic section s on Y of $(K_X \otimes L)|_Y$, then there exists $\tilde{s} \in H^0(X, K_X \otimes L)$ such that

  $$\tilde{s}|_Y = s \quad \text{and} \quad \|\tilde{s}\|_X \leq c\|s\|_Y$$

  for some constant $c > 0$.
- The input L^2 norm $\|s\|_Y$ plays here the crucial role of deciding whether a given section can be extended or not.
- Since [Ohsawa-Takegoshi 1987], there have been extensive developments on L^2 extension, especially when Y is of codimension 1. Our interest is in the generality of Y being of arbitrary codimension.

Application to L^2 extension theorems in SCV

- A previous paper [K. 2007] gave a general L^2 extension theorem formulated for a log-canonical center $Y \subset X$ of a log-canonical pair.
- More recently [Demailly 2015] gave an L^2 extension theorem (essentially) in the same setting from a different viewpoint, taking the input L^2 norm $\|s\|_Y$ w.r.t. **Ohsawa measure** $dV[\Psi]$ on Y. It can be understood in terms of **fiber integral** along $\mu : E \to Y$ (exceptional divisor lying over Y in a log-resolution of the LC pair at hand).
- In [K.2007], the L^2 norm was taken wrt "**Kawamata metric**" defined in terms of the **discriminant divisor** of $\mu : E \to Y'(\to Y)$.
- From our main result, **these L^2 norms are essentially equivalent** and the two theorems (each with its own advantages) can be combined, strengthened.